

Principle

Two identical gravity pendula with a particular characteristic frequency are coupled via a "soft" spiral spring. The amplitudes of both pendula are recorded as a function of time. The coupling factors are determined by way of different methods. Then, the local points of the oscillation are integrated into the video.

Related topics

Spiral spring, gravity pendulum, spring constant, torsional oscillation, torque, beat, angular velocity, angular acceleration, and characteristic frequency

Equipment

2	Pendula with recorder connection	02816-00
1	Helical spring, 3 N/m	02220-00
2	Bench clamp PHYWE	02010-00
2	Support rod PHYWE, square, $l = 630$ mm	02027-55
1	Support rod, stainless steel, 500 mm	02032-00
4	Right angle clamp PHYWE	02040-55
1	Measuring tape, $l = 2$ m	09936-00
1	Dynamometer, transparent, 1 N	03065-02
1	"measure Dynamics" software	14440-61

Additional material

Video camera, tripod, computer



Figure 1: Experimental set-up

Tasks

1. Determination of the spring constant of the coupling spring.
2. Determination and adjustment of the characteristic frequency of the uncoupled pendulum. Determination of the moment of inertia of the pendulum.
3. Graphical representation of the oscillation of the two pendula as a function of time and determination of the oscillation frequency compared to the theoretical oscillation frequency for
 - A) the "in phase" oscillation.
 - B) the "antiphase" oscillation.
 - C) the beat case.

Set-up and procedure

Prior to starting the measurements, the exact value of the spring constant D_F of the coupling spring must be determined. For this purpose the spring is placed on the table, fixed on one side and extended at the other side by means of the dynamometer. In doing so the required force F can be read off the dynamometer and the elongation x is measured with the measuring tape. The spring constant can be easily determined from Hooke's law:

$$D_F = \frac{F}{x}$$

It is recommended to lengthen the spring by 10 cm, 20 cm and 30 cm in order to obtain a mean value for the spring constant D_F .

Set up both pendula without the coupling spring as shown in Figure 1.

In order to cause the pendula to oscillate, touch the rods of the pendula in the upper third with the tips of your finger and displace them simultaneously in the same or opposite directions until the desired amplitude is reached. In this way, transverse oscillations can be avoided. In view of the subsequent experiment, it should be ensured that the pendula oscillate in the same plane.

The videos are used to determine the period of oscillation T_0

for each pendulum. The value of the period of oscillation T of both pendula must be identical. If there are deviations, the lengths of the pendula must be corrected.

For the experiment with the coupling spring, connect the spring to the two plastic hooks on the pendulum rods in two positions. These positions must be equidistant from the pivot of the pendulum.

Measure the amplitudes as a function of time with the following initial conditions:

- A) Both pendula are deflected in the same direction with the same amplitude and they are released simultaneously ("in phase" oscillation).
- B) Both pendula are deflected with the same amplitude, but in opposite directions ("antiphase" oscillation), and they are released simultaneously.
- C) One pendulum remains at rest. The second pendulum is deflected and released (beat case). In this case, satisfactory results can only be achieved if the pendula are properly re-adjusted during the preparation phase so that they actually have the same period of oscillation \bar{T}_0

In all three cases, the oscillations must be recorded for at least one to three minutes. Then, the average values of the corresponding periods of oscillation can be determined based on the plotted curves.

In terms of the video that will be recorded, the following must be taken into consideration concerning the setting and positioning of the camera:

- Set the number of frames per second to approximately 30 fps.
- Select a light-coloured, homogeneous background.
- Provide additional lighting for the experiment.
- The experiment set-up should be in the centre of the video. To ensure this, position the video camera on a tripod centrally in front of the experiment set-up.
- The experiment set-up should fill the video image as completely as possible.
- The optical axis of the camera must be parallel to the experiment set-up.
- For scaling, the length of the pendulum arm must be measured.

Then, the video recording process and the experiment can be started.

Theory

If two gravity pendula P_1 and P_2 with the same characteristic frequency ω_0 are coupled by way of a spring, the following is true for the torque in the case of the position of rest and in the case of small deflections due to gravity and due to the spring tension (see Figure 2):

Torque due to gravity:

$$M_{s,0} = mgL \sin(\Phi_0) \approx mgL \Phi_0 \quad (1)$$

Torque due to spring tension:

$$M_{F,0} = -D_F x_0 l \cos(\Phi_0) \approx -D_F x_0 l$$

D_F	Spring constant
x_0	Elongation of the spring
l	Coupling length
m	Mass of the pendulum
L	Length of the pendulum
g	Gravitational acceleration
Φ_0	Angle between the vertical and the position of rest

If P_1 is deflected by D_F and P_2 by Φ_2 (see Figure 3) and if both pendula are released simultaneously, the following results because of

$$I\ddot{\Phi} = M$$

I = Moment of inertia of the pendulum around its pivot

$$I\ddot{\Phi}_1 = M_1 = -mgL\Phi_1 + D_F l^2 (\Phi_2 - \Phi_1) \quad (2)$$

$$I\ddot{\Phi}_2 = M_2 = -mgL\Phi_2 + D_F l^2 (\Phi_2 - \Phi_1)$$

Following the introduction of the abbreviations

$$\omega_0^2 = \frac{mgL}{I} \quad \text{and} \quad \Omega^2 = \frac{D_F l^2}{I} \quad (3)$$

we obtain the following based on equation (2)

$$\ddot{\Phi}_1 + \omega_0^2 \Phi_1 = -\Omega^2 (\Phi_2 - \Phi_1) \quad (4)$$

$$\ddot{\Phi}_2 + \omega_0^2 \Phi_2 = +\Omega^2 (\Phi_2 - \Phi_1)$$

With, $t=0$ the following three initial conditions are successfully realised.

A) "in phase" oscillation

$$\Phi_1 = \Phi_2 = \Phi_A \quad ; \quad \dot{\Phi}_1 - \dot{\Phi}_2 = 0$$

B) "antiphase" oscillation

$$-\Phi_1 = \Phi_2 = \Phi_A \quad ; \quad \dot{\Phi}_1 - \dot{\Phi}_2 = 2\dot{\Phi}_A \quad (5)$$

C) beat case

$$\Phi_1 = \Phi_A \quad ; \quad \Phi_2 = 0 \quad ; \quad \dot{\Phi}_1 - \dot{\Phi}_2 = \dot{\Phi}_A$$

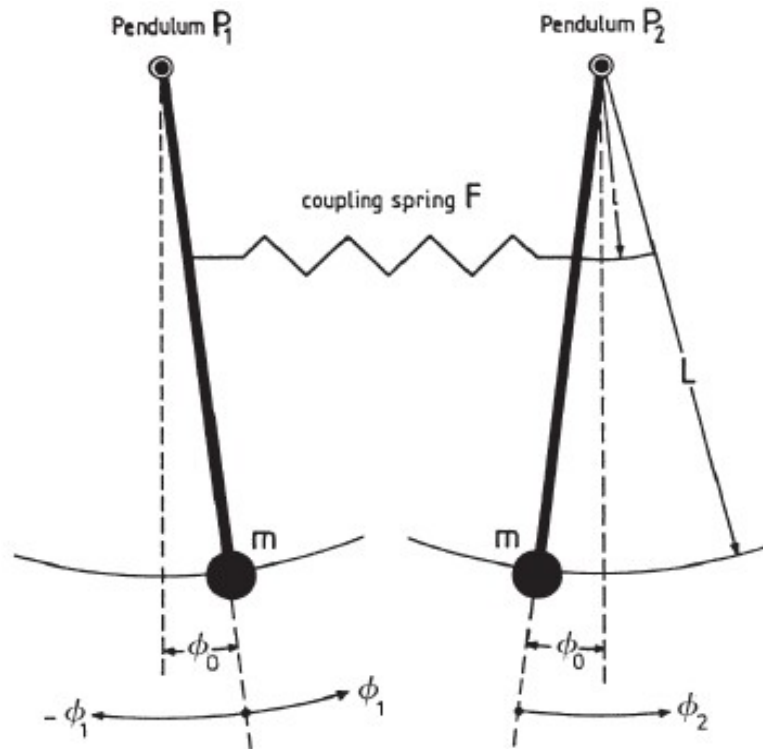


Figure 2: Diagram indicating the various names that are used in the context of coupled pendula

The general solutions of the differential equations (4) with the initial conditions (5) are:

A: $\Phi_1(t) = \Phi_2(t) = \Phi_A \cos(\omega_0 t)$ (6a)

B: $\Phi_1(t) = \Phi_A \cos(\sqrt{\omega_0^2 + 2\Omega^2} t)$ (6b)

$\Phi_2(t) = -\Phi_A \cos(\sqrt{\omega_0^2 + 2\Omega^2} t)$

C: $\Phi_1(t) = \Phi_A \cos\left(\frac{\sqrt{\omega_0^2 + 2\Omega^2} - \omega_0}{2} \cdot t\right) \cdot \cos\left(\frac{\sqrt{\omega_0^2 + 2\Omega^2} + \omega_0}{2} \cdot t\right)$ (6c)

$\Phi_1(t) = \Phi_A \cos\left(\frac{\sqrt{\omega_0^2 + 2\Omega^2} - \omega_0}{2} \cdot t\right) \cdot \sin\left(\frac{\sqrt{\omega_0^2 + 2\Omega^2} + \omega_0}{2} \cdot t\right)$

Note

A) "in phase" oscillation

Both pendula oscillate in phase and with the same amplitude and frequency ω_G . The latter is identical with the characteristic frequency of the ω_0 uncoupled pendulum.

$$\omega_g = \omega_0 \quad (7a)$$

B) "antiphase" oscillation

Both pendula oscillate with the same amplitude and frequency ω_C , but with a phase difference of π . In accordance with (3), the angular frequency

$$\omega = \sqrt{\omega_0^2 + 2\Omega^2} \quad (7b)$$

depends on the length of the pendulum l

C) Beat case

For a weak coupling, e.g., $\omega_0 \gg \Omega$ the angular frequency of the first factor can be expressed as follows:

$$\omega_1 = \frac{\sqrt{\omega_0^2 + 2\Omega^2} - \omega_0}{2} \simeq \frac{\Omega^2}{2\omega_0} \quad (8a)$$

For the angular frequency of the second factor, we get:

$$\omega_2 = \frac{\sqrt{\omega_0^2 + 2\Omega^2} + \omega_0}{2} \simeq \omega_0 + \frac{\Omega^2}{2\omega_0} \quad (8b)$$

As a result, we obtain the following:

$$\omega_1 < \omega_2 \quad .$$

Figure 2 shows the amplitudes $\Phi_1(t)$ and $\Phi_2(t)$ of both pendula as a function of time for the beat case and for different coupling lengths. As the coupling factor, we define the ratio

$$K = \frac{D_F l^2}{mgL + D_F l^2} \quad (9)$$

From equations (3) and (9) we get

$$K = \frac{\Omega^2}{\omega_0^2 + \Omega^2} \quad (10)$$

The coupling factor K from equation (10) can be calculated based on the frequencies of the individual oscillation modes. Substituting equations (7a) and (7b) in equation (10) results in

$$K = \frac{\omega_c^2 - \omega_g^2}{\omega_c^2 + \omega_g^2} \quad (11)$$

("antiphase" oscillation)

Substituting equations (8a) and (8b) in equation (10) results in

$$K = \frac{2\omega_1\omega_2}{\omega_1^2 + \omega_2^2} \quad (12)$$

(beat case)

In order to test the influence of the coupling length on the frequencies of the individual oscillations, we substitute equations (11) and (12) in equation (9). This yields the following for the "antiphase" oscillation:

$$\omega_1^2 = \frac{2D_F\omega_0^2}{mgL}l^2 + \omega_0^2 \quad (13)$$

The following results for the beat case:

$$\omega_1 = \omega_0 \frac{D_F}{2mgL}l^2 \quad (14)$$

and

$$\omega_2 = \omega_0 \frac{D_F}{2mgL}l^2 + \omega_0 \quad (15)$$

Evaluation

Task 1: Determination of the spring constant of the coupling spring.

Transfer the video to the computer. Then, start "measure Dynamics" and open the video under "File" – "Open video ...". Mark the length of the spring with the scale that appears in the video by way of "Video analysis" – "Scaling ..." – "Calibration" and enter the length of the spring without a load (in this case 14 cm) into the input window. Next, create three new columns via the table menu line. Enter the mass of the weights (name: "Mass", unit: "kg" or "g") into the first column, the length of the spring (name: "L", unit: "m") into the second column, and the elongation of the spring (name: "Elongation", unit: "m", formula: "L-0.14") into the third column. Next, open "Measure length" under "Measure". Then, fast forward the video to the points where additional weights are suspended from the spring. Enter the mass and determine the length of the spring in the same line (the scale that appears in the video is laid along the spring) and enter it also. Select "Display" and "Diagram", click "Options", delete all of the already existing graphs, and select the graphs "Mass" (horizontal axis) – "Elongation" (vertical axis). This leads to:

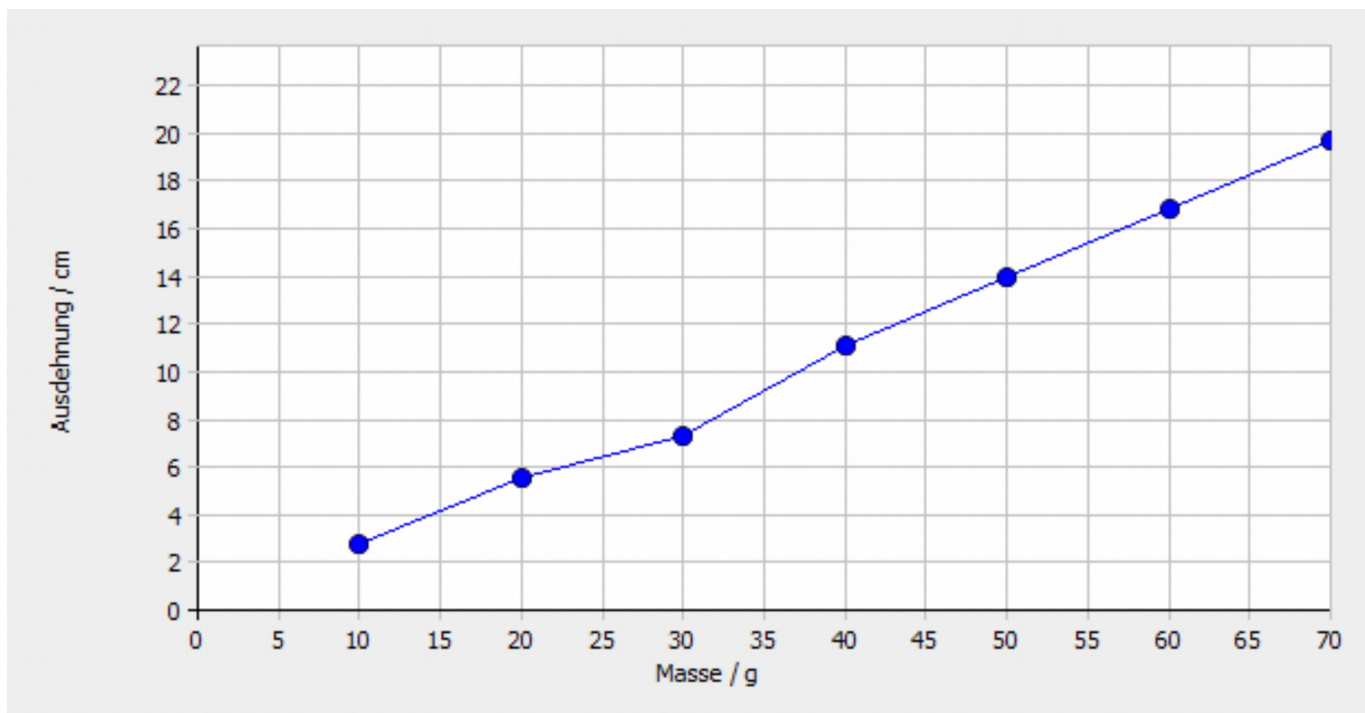


Figure 3: Representation of the elongation of the spring as a function of the mass to which it is subjected

Figure 3 shows the linear relationship between the mass and the elongation. Clicking "Options" in the menu line of the diagram and selecting the tab "Linear regression" will add a regression line to the diagram and the corresponding function will be displayed in the menu window. In this case here, the gradient of the line is 0.2862. This means that the spring constant D_F is: 2.86 N/m.

Task 2: Determination and adjustment of the characteristic frequency of the uncoupled pendulum. Determination of the moment of inertia of the pendulum.

Transfer the video that has been recorded to the computer. Then, start "measure Dynamics" and open the video under "File" - "Open video ...". Mark the start of the experiment ("Start selection" and "Time zero") and the end of the experiment ("End selection") in the video for further analysis via the menu line above the video. The experiment begins with the deflection of the pendulum and it ends after several pendulum oscillations. For this experiment, it is important to ensure that the experiment ends after a specific number of oscillations. Determine the number of oscillations and read off the time that was needed for these oscillations.

The following results for pendulum (1):

$$T_1 = 1.996 \text{ s}$$

or

$$\omega_1 = \frac{2\pi}{T_1} = 3.15 \cdot \frac{1}{\text{s}}$$

The following results for pendulum (2):

$$T_2 = 1.996 \text{ s}$$

or

$$\omega_2 = \frac{2\pi}{T_2} = 3.15 \cdot \frac{1}{\text{s}}$$

This means that the periods of oscillation of the two pendula are identical. If this is not the case, the pendulum lengths must be corrected for the subsequent experiments until the periods of oscillation are identical.

Based on equation (3) and the known pendulum mass $m = 1 \text{ kg}$ of, the following results for the moment of inertia of the pendulum:

$$I_{1/2} = 0.978 \text{ kg} \cdot \text{m}^2$$

Task 3: Graphical representation of the oscillation of the two pendula as a function of time and determination of the oscillation frequency compared to the theoretical oscillation frequency for

(A) *the "in phase" oscillation.*

Transfer the recorded videos to the computer in the same manner as in task 2 and specify the time zero as well as the start and end of the experiment. Then, mark the pendulum arm with the scale that appears in the video by way of "Video analysis" – "Scaling ..." – "Calibration" and enter the length that has been measured beforehand into the input window. In addition, enter the frame rate that has been set for the recording process under "Change frame rate" and position the origin of the system of coordinates in the pivot of the pendulum under "Origin and direction". Turn the system of coordinates by right-clicking so that the x-axis points horizontally to the left.

Then, the actual motion analysis can be started under "Video analysis" – "Automatic analysis" or "Manual analysis". For the automatic analysis, we recommend selecting "Motion and colour analysis" on the "Analysis" tab. Under "Options", the automatic analysis can be optimised, if necessary, e.g. by changing the sensitivity or by limiting the detection radius. Then, look for a film position in the video where the object that is to be analysed is perfectly visible. Click the object. If the system recognises the object, a green rectangle appears and the analysis can be started by clicking "Start".

If the automatic analysis does not lead to any satisfying results, the series of measurements can be corrected under "Manual analysis" by manually marking the object that is to be analysed.

In this experiment, two motions, i.e. the motions of the two pendula, must be analysed in one video. To do so, we recommend using a separate worksheet for each motion, i.e. for each analysis. You can switch between the worksheets in the table menu line. First, analyse the motion of the first pendulum, followed by the analysis of the motion of the second pendulum in a second worksheet. For the second analysis, however, the position of the origin of the system of coordinates must be changed. Move it into the pivot of the second pendulum.

Since the pendulum oscillates in a plane, we recommend adding another column to the worksheets via the table menu line. This new column will hold the angle of deflection φ . The column must be named as follows: Name: "Angle", unit: "°", formula: " $\arctan2(x;y)*360/(2*\pi)$ ".

For the graphical representation, open "Display" and "Diagram", click "Options", delete all of the already existing graphs, select the graphs t (horizontal axis) – "Angle" (vertical axis) from both worksheets, and click "Add". This leads to:

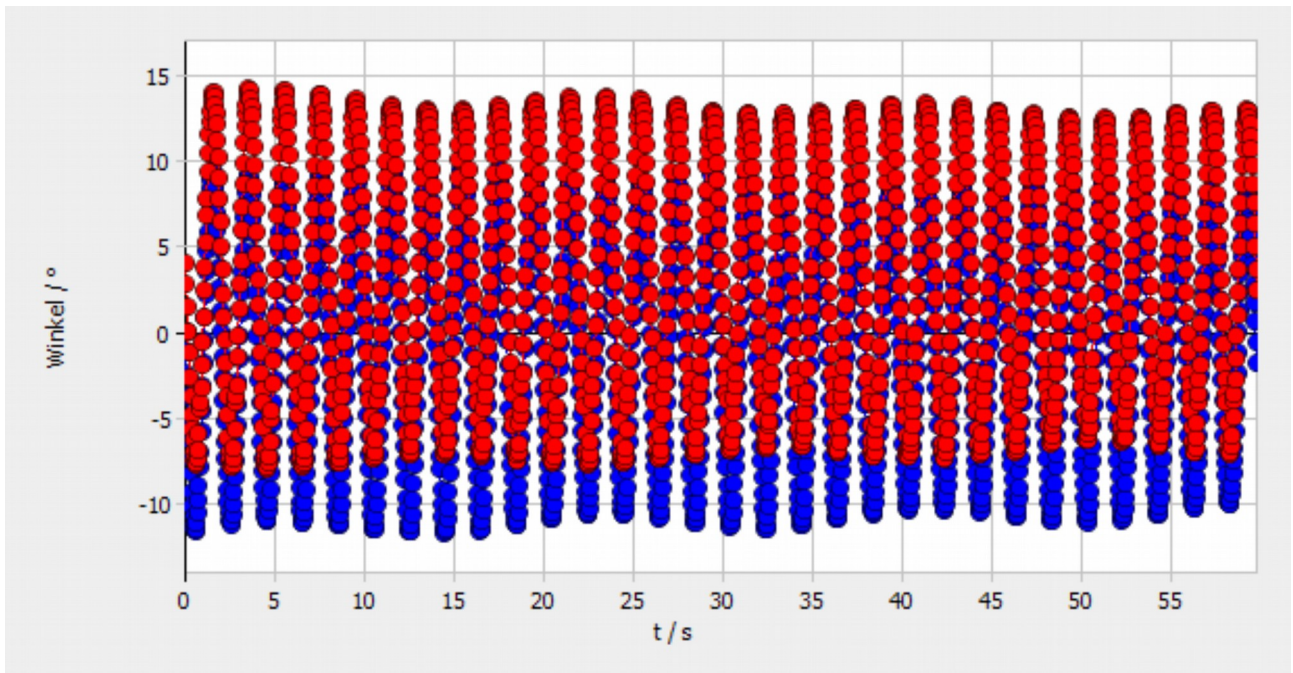


Figure 4: Representation of the deflection of both pendula for the "in phase" oscillation as a function of time t

If we look at a smaller detail, the following results:

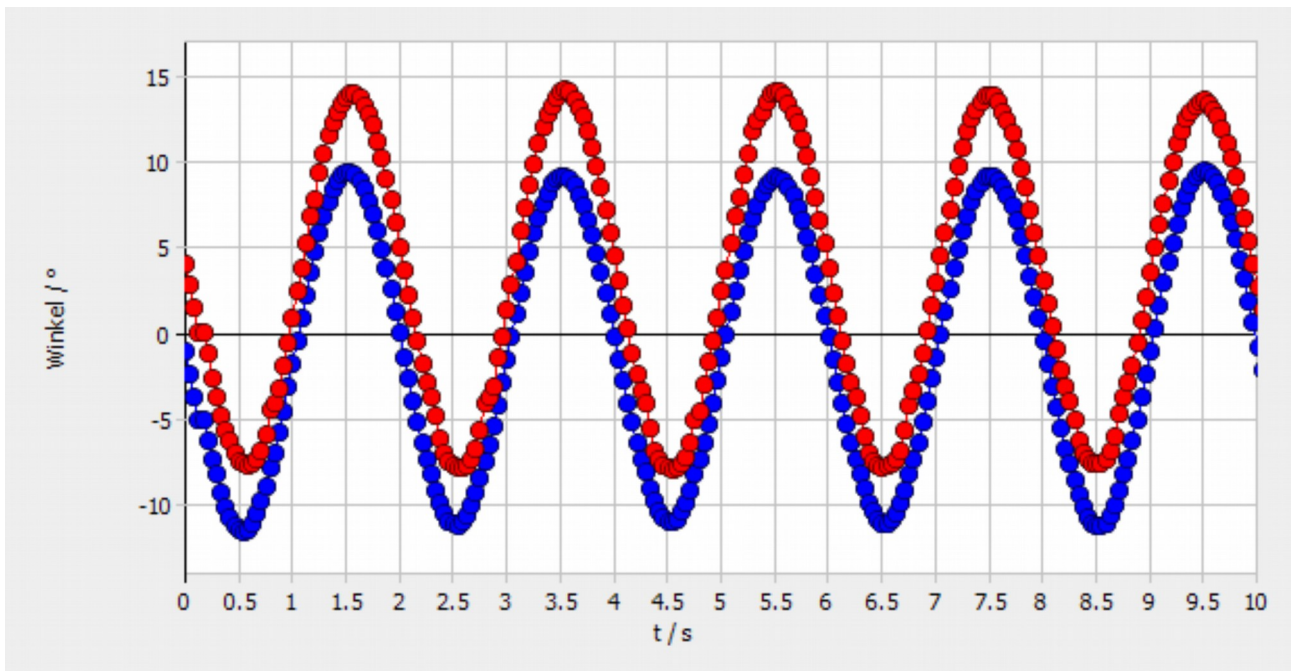


Figure 5: Detail of the deflection of both pendula for the "in phase" oscillation as a function of time t

Figure 5 shows that both pendula oscillate with the same amplitude and frequency ω_g . The y -axis is offset, since the oscillation is not symmetrical with regard to the zero position without a coupling spring. The frequency ω_g is:

$$\omega_g = \frac{2 \cdot \pi}{T} = \frac{2 \cdot \pi}{1.993 \text{ s}} = 3.15 \cdot \frac{1}{\text{s}}$$

This value corresponds to the value of the characteristic frequency of the uncoupled pendulum. As a result, the theory has been confirmed.

(B) "antiphase" oscillation

For the "antiphase" oscillation, the process is the same as for A). This leads to:

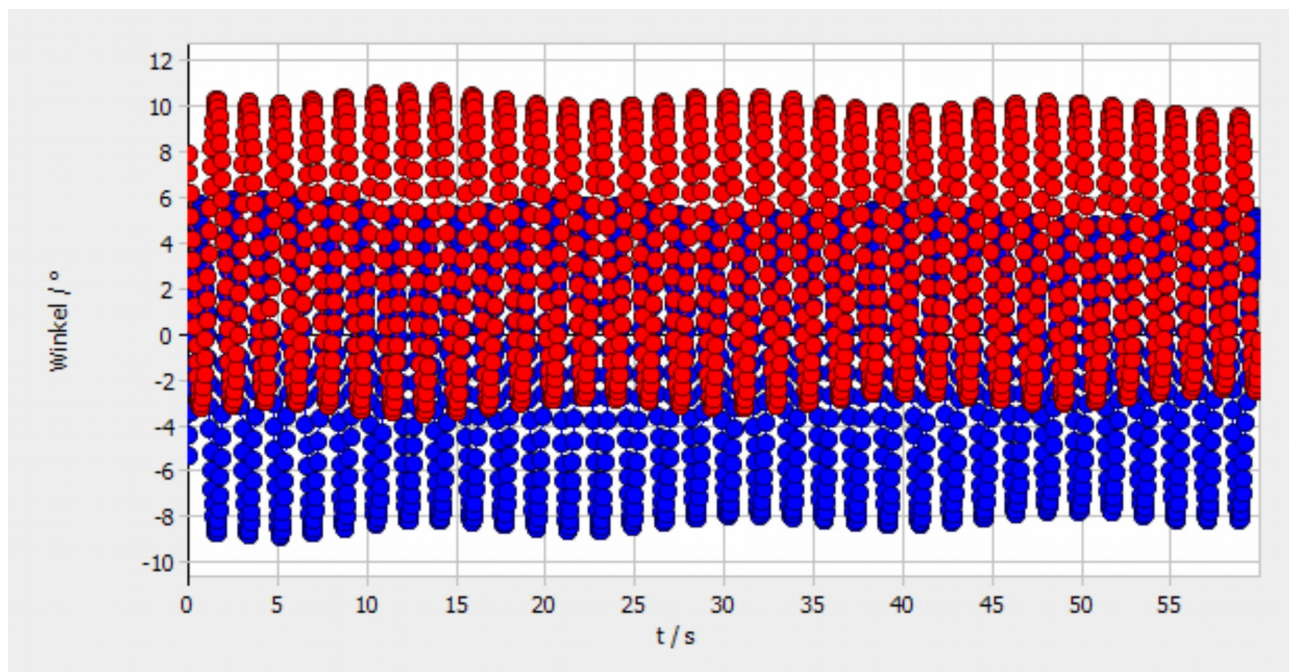


Figure 6: Representation of the deflection of both pendula for the "antiphase" oscillation as a function of time t

If we look at a smaller detail, the following results:

Figure 7 shows that both pendula oscillate with the same amplitude and frequency ω_c , but with a phase difference of π .

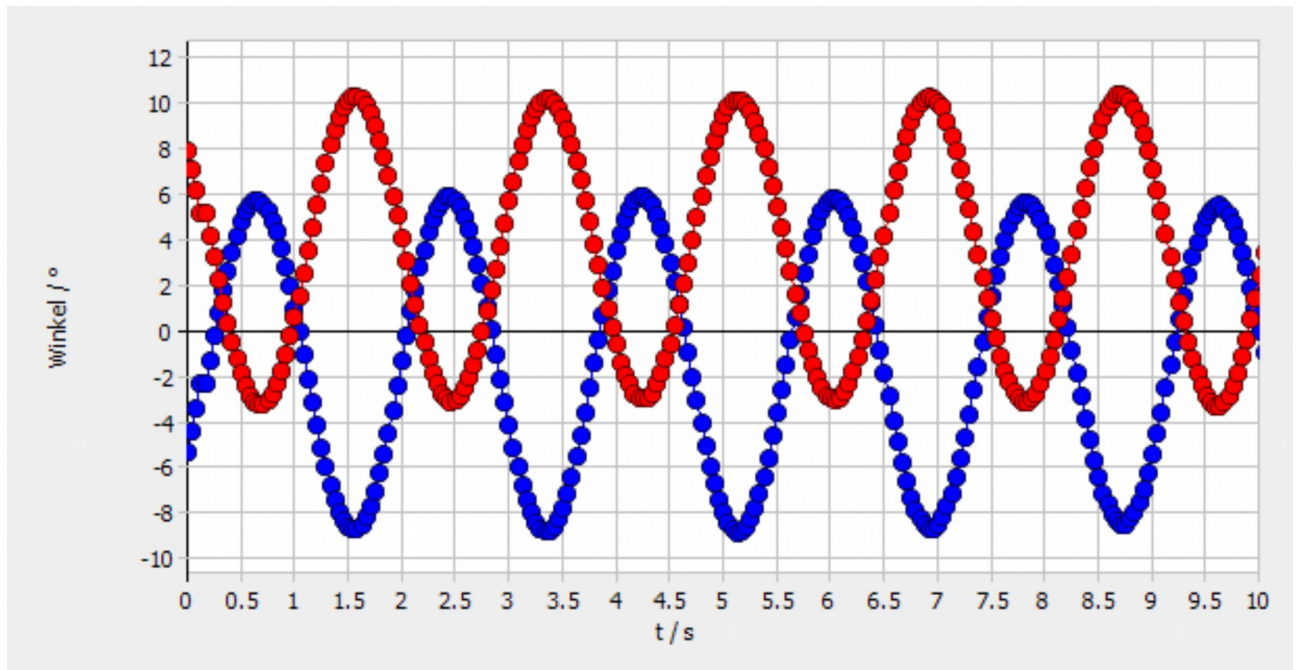


Figure 7: Detail of the deflection of both pendula for the "antiphase" oscillation as a function of time t

The frequency is:

$$\omega_c = \frac{2 \cdot \pi}{T} = \frac{2 \cdot \pi}{1,789 \text{ s}} = 3.51 \cdot \frac{1}{\text{s}}$$

Based on equation (13), the theoretical frequency is:

$$\omega_c = 3.47 \cdot \frac{1}{\text{s}}$$

As a result, the theory has been confirmed.

(C) *beat case*

The process for the "beat" case is the same as for A) and B). This leads to:

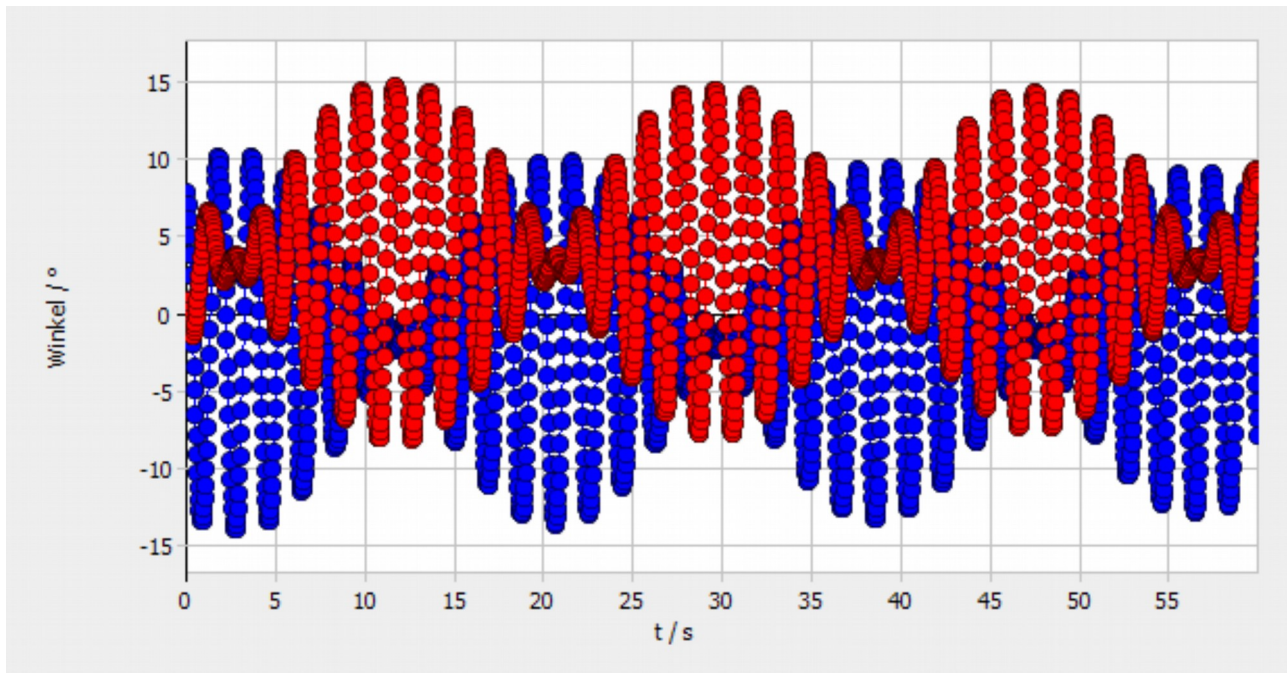


Figure 8: Representation of the deflection of both pendula for the "beat" case as a function of time t

If we look at a smaller detail, the following results:

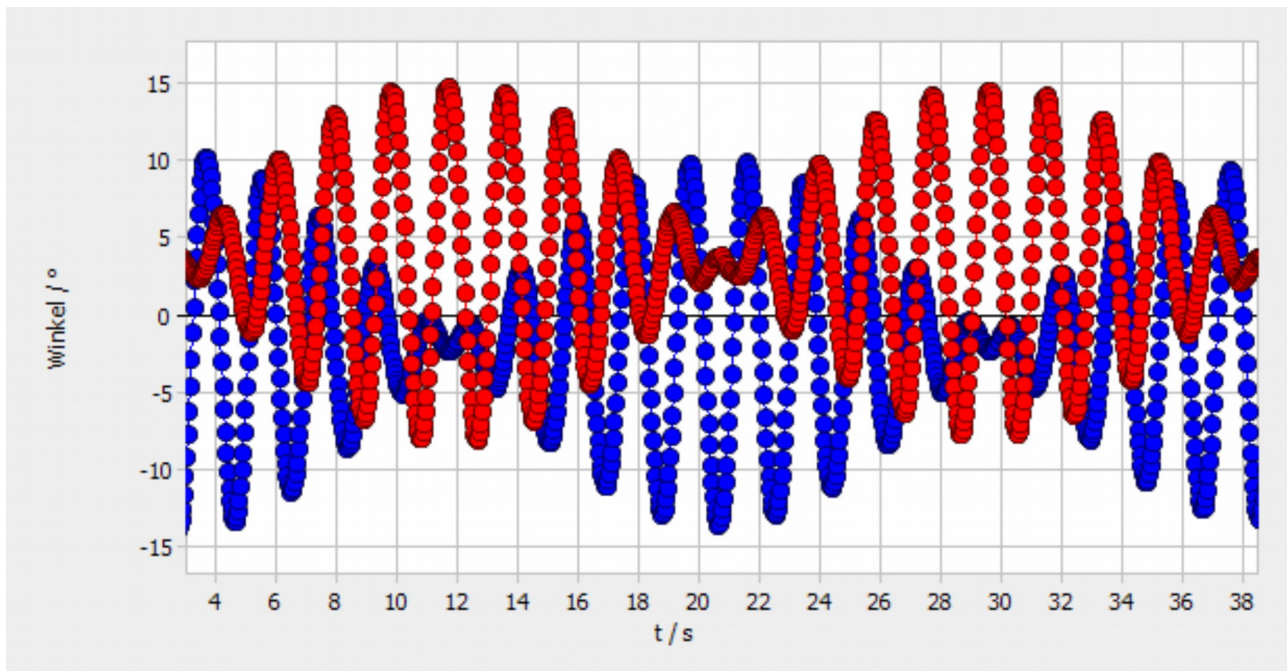


Figure 9: Detail of the deflection of both pendula for the "beat" case as a function of time t

Figure 9 shows that there are two oscillations: the to and fro oscillation of the pendulum with the frequency P_1 and the reciprocal transfer of the amplitude with the frequency ω_2 . They are:

$$\omega_1 = \frac{2 \cdot \pi}{T_1} = \frac{2 \cdot \pi}{35.87 \text{ s}} = 0.175 \cdot \frac{1}{\text{s}},$$

$$\omega_2 = \frac{2 \cdot \pi}{T_2} = \frac{2 \cdot \pi}{1.789 \text{ s}} = 3.51 \cdot \frac{1}{\text{s}}.$$

Based on the equations (8a) and (8b) or (14) and (15), respectively, the theoretical frequencies are:

$$\omega_1 = 0.169 \cdot \frac{1}{\text{s}},$$

$$\omega_2 = 3.32 \cdot \frac{1}{\text{s}}.$$

The experimental frequency values are nearly identical with the theoretical, calculated values.